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Mass Parameter Quantization in the FRW Cosmological Model

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Using the canonical quantum theory apply to spherically symmetric pure gravitational systems, we present the study of the closed Friedmann-Robertson-Walker (FRW) cosmological model filled with pressureless matter (dust) content as a toy model. The Wheeler-DeWitt equation is view as the Schrödinger equation for the linear harmonic oscillator with energy E. We show that such type of universe has a quantized masses of the order of the Planck mass and harmonic oscillator wave functions, where a dual symmetry emerge among the quantum parameters.

KEY WORDS: quantum gravity; quantum cosmology; dual simmetry.

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1. INTRODUCTION

The absence of a fundamental understanding of physics at very high energies and, in particular, in the absence of a consistent quantum theory of gravity, there is no hope, at present, to meet an understanding of the quantum origin of the Universe in a definitive way. However, it appears desirable to develop highly and simplified, but consistent toy models, which contain as many as possible of those features which are believed will be present in a future complete quantum theory of gravity.

In order to give one possible mechanism for to find certain quantization rules for the parameters that describes our universe, we present the simpler approach, where we view the Wheeler-DeWitt equation at energy zero, obtained with the canonical procedure quantization for a closed FRW cosmological model filled with pressureless matter (dust) content, like the Schrödinger equation for a linear harmonic oscillator at energy E. This energy is associated with the mass parameter quantization, and such type of universe has a quantized masses of the order of

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the Planck mass and harmonic oscillator wave functions with all properties in the usual sense.

Some time ago, Rosen (1993) used the equations of General Relativity for the case of a closed homogeneous isotropic universe, the equation obtained was like that for the s-state of a hydrogen-like atom, and was able to obtain the relation $m_n = \sqrt{n\pi} M_{\rm pl}$ for the quantization of the mass spectrum.

But, the main work in mass quantization rule, yield in the context of black hole. In the framework of quantum gravity, black holes must be treated as quantum objects. As such, they are characterized by quantum numbers like mass, electric charge and angular momentum. For neutral, non-rotating Schwarzschild black hole, the only quantum number which is left is the mass M. Classically, it is related to the area A of the black hole horizon by the relation

$$A = \frac{16\pi G^2 M^2}{c^4}, \tag{1}$$

where G is the Newtonian gravitational constant and c is the velocity of light in vacuum. Important questions in black holes physics are what the spectrum of A looks like and what the degeneracies of states are for a given values of A. Following Eq. (1), any change in the mass parameter, imply one change in the area of the black hole horizon.

The quantization of black holes was first proposed by Bekenstein some years ago (Bekenstein, 1974) and recently Cavaglia *et al.* (1995, 1996), using an hamiltonian formalism for black hole, develop a canonical formalism in the radial variable r that is timelike inside the Schwarzschild horizon. The fundamental idea of Bekenstein's work is the remarkable observation that the horizon area of a non-external black hole behaves as a classical adiabatic invariant. But in the spirit of Ehrenfest principle (Ehrenfest, 1959), any classical adiabatic invariant should correspond to a quantum entity with discrete spectrum. Bekenstein conjectured that the horizon area of a quantum black hole should have a discrete spectrum with uniformly spaced eigenvalues of the form

$$A_n = \gamma l_{\rm pl}^2 n, \quad n = 1, 2, 3, \tag{2}$$

where γ is a dimensionless constant to be determined, and $\ell_{\rm pl} = (\frac{G\hbar}{{\rm c}^3})^{1/2}$ is the Planck length. Bekenstein's proposal implies that the energy eigenvalues corresponding to the stationary states of the black holes are

$$E_n = \sigma\sqrt{n} E_{\rm pl}, \quad n = 1, 2, \ldots, \quad E_{\rm pl} = \sqrt{\frac{\hbar c^5}{G}}, \tag{3}$$

where $\sigma = \sqrt{\frac{\gamma}{16\pi}}$ is the order of unity.

On the other hand, the quantum theory of gravity has given rather a few direct physical predictions, perhaps the most important of them are the existence of the so called Hawking radiation emitted by the black hole (Hawking, 1975) and the result given by Ashtekar, Rovelli and Smolin, which says that area is quantized (Ashtekar *et al.*, 1992)

Using a combination of thermodynamics and statistical physics arguments it was found by Bekenstein and Mukhanov (1995), Mukhanov (1986) that the dimensionless constant γ should be of the form $\gamma = 4/n$ In α, where α is the degeneracy factor of the nth area level. Recently, Hod (1998) employed Bohr's correspondence principle and found evidence in favor of the value $\alpha = 3^n$. An analogous scenario appears in our model.

The quantum solution of the FRW cosmological model has been calculated in many works (Padmanabhan, 1983a,b; Socorro, 2003; Socorro *et al.*, 2003), but not related to mass quantization.

The main purpose of this work is to obtain a time independent Schrödinger equation for the case of the closed FRW model, where dust matter is filling the universe, and obtain a mass spectrum for particular model. This is done following the canonical quantization procedure by means of which, we show that our system looks (with good approximation) like a linear quantum oscillator. Also, we obtain the wave function of the FRW cosmological model, in this context. In the Mäkelä's work (Mäkelä, 1986), the author describe the gravitational degrees of freedom of the Schwarzschild black hole by one free variable, introducing an equation which suggest to be the time independent Schrödinger equation of the Schwarzschild black hole that is similar to the one in our toy model.

The remainder of the paper is organized as follow. In Section II, using the canonical formalism, we construct the corresponding Hamiltonian for the FRW cosmological model, In Section III, the time independent Schrödinger equation is obtained, promoving the classical Hamiltonian to operators, and applying it to the wave function ψ, $\hat{\mathrm{H}}\psi = 0$. Here we introduce the quantization rules for the energy, which depends on an integer number n. These quantization rules were obtained using the creation-annihilation representation. In Section IV, the mass spectrum is calculated. Finally, Section V is devoted for conclusions.

2. THE CANONICAL HAMILTONIAN

Observations shown that our universe is homogeneous ans isotropic with very good approximation. Theoretically, we say that the cosmological principle is valid. This homogeneous and isotropic space-time was originally studied by Friedmann, Robertson, and Walker (FRW). The symmetry is encoded in the special form of following line element

$$ds^2 = -N^2(t)dt^2 + R^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right] \tag{4}$$

where $R(t)$ is the scale factor, $N(t)$ the lapse function, κ is the constant curvature, taking the values 0, $+1$, -1 (flat, closed and open space, respectively).

In this work, we consider the classical lagrangian for a pure gravity system and the corresponding term of matter content, perfect fluid with barotropic state equation $p = \gamma\rho$, and cosmological term (Socorro, 2003; Socorro *et al.*, 2003)

$$L = -\frac{c^2 R}{2NG}\left(\frac{dR}{dt}\right)^2 + N\frac{\kappa c^4}{2G}R + N\frac{c^4\Lambda}{6G}R^3 - NM_\gamma c^2 R^{-3\gamma}. \tag{5}$$

following the canonical procedure, we obtaining the classical Hamiltonian,

$$\mathcal{H} = -\frac{1}{2R}\Pi_R^2 - N\frac{\kappa c^4}{2G}R - N\frac{c^4\Lambda}{6G}R^3 + NM_\gamma c^2 R^{-3\gamma}. \tag{6}$$

Considering the dust case $\gamma = 0$, $\kappa = 1$ and $\Lambda = 0$, we obtain

$$S = \int\left[-\frac{c^2}{2GN}R\dot{R}^2 + \frac{c^4}{2G}NR - NE_s\right]dt. \tag{7}$$

with $E_s = Mc^2$, where the parameter M corresponds to the mass parameter of the closed Universe and dust scenario.

The action (7) preserves the invariance under time reparametrization

$$\delta t = a(t), \tag{8}$$

if the transformations of $N(t)$ and $R(t)$ are defined as

$$\delta N = \frac{d(aN)}{dt}, \quad \delta R = a\frac{dR}{dt}. \tag{9}$$

Note that if we take the lapse function as

$$N(t) = \tilde{N}(t)R(t)\frac{c^2}{M_{\rm pl}G}, \tag{10}$$

and substituting into (7), we have the following invariant action

$$S = \int\left[-\frac{M_{\rm pl}}{2N}\dot{R}^2 + \frac{c^6}{2M_{\rm pl}G^2}\tilde{N}R^2 - \tilde{N}\frac{Mc^4}{M_{\rm pl}G}R\right]dt. \tag{11}$$

Using the relations (9,10), it is easy to show that $\tilde{N}(t)$ transforms as

$$\delta\tilde{N} = \frac{d(a\tilde{N})}{dt}. \tag{12}$$

Proceeding with the Hamiltonian analysis, we define the usual canonical momentum conjugate to the $R(t)$ coordinate, $P_R = \frac{\partial L}{\partial R}$ and performing the Legendre transformation, we can obtain the following canonical Hamiltonian

$$H_{\text{can}} = \tilde{N}\left[-\frac{P_R^2}{2M_{\text{pl}}} - \frac{c^6}{2M_{\text{pl}}G^2}R^2 + \frac{M}{M_{\text{pl}}}\frac{c^4}{G}R\right]$$
$$= \tilde{N}\left[-\frac{P_R^2}{2M_{\text{pl}}} - \frac{M_{\text{pl}}}{2}\omega_0^2\left(R - \frac{MG}{c^2}\right)^2 + \frac{M}{2M_{\text{pl}}}Mc^2\right], \tag{13}$$

where $\omega_0 = \frac{c^3}{M_{\text{pl}}G}$ is the fundamental frequency of the system. This form of the canonical Hamiltonian explains the fact, that the lapse function $\tilde{N}(t)$ is a Lagrange multiplier, which enforces the first class constraint $H = 0$. The latter manifests the invariance of the action under reparametrization transformations (8,9). According to the Dirac's constraint quantization procedure, the wave function must be annihilated by the operator version of the classical constraint, obtaining the corresponding Wheeler-DeWitt equation at zero energy.

We transform Eq. (13) by defining

$$\xi = R - \frac{MG}{c^2}, \tag{14}$$

thus its momentum conjugate becomes $P_\xi = P_R$ and the constraint at the classical level reads as follows

$$H_{\text{can}} = \tilde{N}H = \tilde{N}\left[-\frac{P_\xi^2}{2M_{\text{pl}}} - \frac{M_{\text{pl}}}{2}\omega_0^2\xi^2 + \frac{M}{2M_{\text{pl}}}Mc^2\right] = 0, \tag{15}$$

that can also be rewritten as

$$\frac{P_\xi^2}{2M_{\text{pl}}} + \frac{M_{\text{Pl}}\omega_0^2}{2}\xi^2 = \frac{M}{M_{\text{pl}}}\frac{Mc^2}{2} = \frac{M}{M_{\text{pl}}}\frac{E_s}{2}. \tag{16}$$

3. HARMONIC OSCILLATOR EQUATION AND QUANTIZATION RULES

Making the usual realization of the operator $\frac{\hat{P}_\xi^2}{2M_{\text{Pl}}} = -\frac{\hbar^2}{2M_{\text{Pl}}}\frac{d^2}{d\xi^2}$ and applying it to the wave-function ψ, we get the following linear harmonic oscillator equation

$$\left[-\frac{\hbar^2}{2M_{\text{pl}}}\frac{d^2}{d\xi^2} + \frac{M_{\text{Pl}}\omega_0^2}{2}\xi^2\right]\psi = \frac{M}{M_{\text{pl}}}\frac{E_s}{2}\psi. \tag{17}$$

In this point we make the transformation

$$E_s = \frac{c^4}{2G}R_{\text{sup}} \tag{18}$$

y considering the form of E_s given in (7) we obtain that $R_{\rm sup} = \frac{2MG}{c^2}$, being the radius for the closed universe. Making the transformation $\frac{\xi}{\ell_{\rm pl}} = x$, one can rewrite (17) as

$$\frac{1}{2}\left[x^2 - \frac{d^2}{dx^2}\right]\psi = \frac{1}{4}\frac{R_{\rm sup}E_s}{\ell_{\rm pl}E_{\rm pl}}\psi. \tag{19}$$

Using the creation-annihilation representation,

$$a = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right), \tag{20}$$

$$a^\dagger = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right), \tag{21}$$

with the usual algebra between them, $[a, a^\dagger] = 1$, we can rewrite Eq. (19) as

$$\begin{aligned} a^\dagger a\psi &= \frac{1}{2}\left[x^2 - \frac{d^2}{dx^2}\right]\psi - \frac{1}{2}\psi = \left(-\frac{1}{2} + \frac{1}{4}\frac{R_{\rm sup}E_s}{\ell_{\rm pl}E_{\rm pl}}\right)\psi = n\psi, \\ n &= 0, 1, 2, \ldots. \end{aligned} \tag{22}$$

In this way, we have the following useful relations

$$\begin{aligned} R_{\rm sup}E_s &= r\left(n + \frac{1}{2}\right)\ell_{\rm pl}E_{\rm pl} \\ &= 4\left(n + \frac{1}{2}\right)\hbar c, \end{aligned} \tag{23}$$

$$E_s^2 = 2\left(n + \frac{1}{2}\right)E_{\rm pl}^2, \tag{24}$$

$$\frac{E_s}{2} = \left(n + \frac{1}{2}\right)\hbar\omega_0. \tag{25}$$

One can see that when n is bigger, we find

$$\frac{R_{\rm sup}}{\ell_{\rm pl}} = 2\sqrt{2n+1}. \tag{26}$$

in sense that, when $n \to \infty$, $R_{\rm sup}$ coincide with the maximum expansion in the scale factor R.

Let us write the Eq. (19) in the following form

$$\frac{d^2\psi}{dx^2} + \left(\alpha_n^2 - x^2\right)\psi = 0, \tag{27}$$

where α_n is parameter associated with the energy of the nth eigenstate

$$\alpha_n^2 = \frac{1}{2}\frac{R_{\mathrm{sup}} E_s}{\ell_{\mathrm{pl}} E_{\mathrm{pl}}} = 2\left(n + \frac{1}{2}\right), \quad \text{thus,} \quad \alpha_n = \frac{E_s}{E_{\mathrm{pl}}}, \tag{28}$$

and the quantum solution is similar to the harmonic oscillator case

$$\psi_n(x) = \left(\frac{1}{\sqrt{\pi} n! 2n}\right)^{\frac{1}{2}} H_n(x) e^{-\frac{1}{2}x^2}, \tag{29}$$

with $H_n(x)$ the Hermite polynomials (where an approximation is made on the boundary conditions, see below).

Analyzing the quantum solution (29) in the variable $x = \frac{\xi}{\ell_{\mathrm{pl}}} = \frac{1}{\ell_{\mathrm{pl}}}(R - \frac{MG}{c^2})$, the classical allowed region is the one in which

$$-\frac{R_{\mathrm{sup}}}{2\ell_{\mathrm{pl}}} \leq x \leq \frac{R_{\mathrm{sup}}}{2\ell_{\mathrm{pl}}}, \tag{30}$$

and using (26), the coordinate x have the following range $(-\infty, \infty)$ when $n \to \infty$. For this range in the variable x, the system is really the linear harmonic oscillator with all its properties, i.e, we are considering the boundary conditions

$$\begin{aligned} R = 0 \quad &\to \quad \psi\left(x = -\frac{R_{\mathrm{sup}}}{2\ell_{\mathrm{pl}}} \sim -\sqrt{2n} \sim -\infty\right) = 0, \\ R = R_{\mathrm{max}} = R_s \quad &\to \quad \psi\left(x = \frac{R_{\mathrm{sup}}}{2\ell_{\mathrm{pl}}} \sim \sqrt{2n} \sim \infty\right) = 0, \end{aligned} \tag{31}$$

it is say, that $-\sqrt{2n}$ corresponds to big bang and $\sqrt{2n}$ to the maximum expansion of the universe, for $n \to \infty$. Also, when $\xi = 0$, (see Eq. (14)), $R = \frac{MG}{c^2}$, one half of the R_{sup} radius, the wave function is not zero, it is constant, for few values in the parameter n. This result differs considerably with the Mäkelä's paper (Mäkelä, 1996), because their wave function does not satisfy the usual boundary conditions of the linear oscillator.

4. THE DISCRETE MASS SPECTRUM

Now, it is clear that the system, even in its lowest energy state $n = 0$, has a finite, minimal energy. Equation (23) implies the following quantization mass rule

$$M_n = \sqrt{2n + 1} M_{\mathrm{pl}}. \tag{32}$$

In this point, we introduce the condition on the M_n parameter when $n \to \infty$, this parameter must be the classical mass parameter M_{sup}, for the closed universe, filled with dust matter, in the maximum expansion.

On the other hand, the Eq. (24) is the equivalent relation of (3). In this way, the universe of this type has a quantized mass of the order of the Planck mass $M_{\mathrm{pl}} = 2, 18 \times 10^{-8}$ Kg. These results are similar to those obtained by other methods in the black hole scenario (Kastrup, 1996, 1997; Louko-Mäkelä, 1996; Mäkelä, 1996, 1997; Rosen, 1993).

The difference in mass between any two consecutive eigenvalues is given by

$$\Delta M_{n+1} \equiv M_{n+1} - M_n = \left[\sqrt{1 + \frac{2}{2n+1}} - 1\right] M_n \quad n \text{ finite} \tag{33}$$
$$= 0 \quad n \to \infty,$$

the result when $n \to \infty$ is in agreement with the correspondence principle.

If one goes over from mass to energy units one finds the following for the ground state energy $M_0 c^2 = 1, 22 \times 10^{28}$ eV, and for the excitation to the next state, the energy required will be $(M_1 - M_0)c^2 = 0, 9 \times 10^{28}$ eV.

In general, using the Eq. (33), we have for any two neighbourhoods states

$$\Delta M_{n+1} c^2 = \left[\sqrt{1 + \frac{2}{2n+1}} - 1\right] M_n c^2 \tag{34}$$

On the other hand, we can see that Eq. (23) remains invariant under the dual symmetries,

$$E_s \to \frac{c^4}{2G} R_s, \quad R_{\mathrm{sup}} \to \frac{2G}{c^4} E_s, \tag{35}$$

in analogy with the case of magnetic and electric charges, found by Montonen and Olive (1997).

If we associate an area parameter $A = 4\pi R_{\mathrm{sup}}^2$ to the system, it has corrections depending on the nth eigenstate (see Eq. (2)),

$$A = 2\pi \left[R_{\mathrm{sup}}^2 + \left(\frac{2G}{c^4}\right)^2 E_s{}^2\right] = 32\pi \left(n + \frac{1}{2}\right) \ell_{\mathrm{pl}}^2. \tag{36}$$

The "area of the closed universe" can take only discrete values, such that, the quanta of the area is in the same order of magnitude as the Planck area. It is easy to check that this parameter is invariant under the transformation (35) and looks like as Eq. (2).

5. CONCLUSIONS

In this paper using the canonical quantization, a time-independent Schrödinger equation for the closed FRW cosmological model, given us interesting results, by instants, the area of the ground state ($n = 0$) is proportional to ℓ_{pl}^2, and the higher state corresponds at classical one (see Eqs. (26, 36)) with the corresponding change in the mass parameter, in agreement with the correspondence principle.

This toy model system looks like a quantum linear harmonic oscillator, and using the creation-annihilation representation we found interesting relations between the quantities $R_{\rm sup}$, E_s, $E_{\rm pl}$ and $\ell_{\rm pl}$, (see (23,24,25)), in terms of the discrete parameter n. With these relations, we obtained the discrete mass spectrum for this type of Planck scale closed universe (32). When the eigenvalue n tends at infinite, the parameters that depends of it, by the correspondence principle, will correspond to classical ones. We have the hope that in a more fundamental energy level exist an exact symmetry for all parameter for this toy model, for this reason the supersymmetric generalization of our approach is outlined.

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